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Every planar graph with the Liouville property is amenable

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Abstract

We introduce a strengthening of the notion of transience for planar maps in order to relax the standard condition of bounded degree appearing in various results, in particular, the existence of Dirichlet harmonic functions proved by Benjamini & Schramm. As a corollary we obtain that every planar non-amenable graph admits non-constant Dirichlet harmonic functions.

1 Introduction

A well-known result of Benjamini & Schramm [6, 7] states that every transient planar graph with bounded vertex degrees admits non-constant harmonic functions with finite Dirichlet energy; we will call such a function a (non-constant) *Dirichlet harmonic function* from now on. In particular, such a graph does not have the Liouville property. Two independent proofs of this theorem were given in [6, 7], one using circle packings and one using square tilings.

The bounded degree condition was essential in both these proofs, and is in fact necessary: consider for example a 1-way infinite path where the nth

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edge has been duplicated by 2^n parallel edges. Still, there are natural classes of unbounded degree graphs where such obstructions do not occur, and it is interesting to ask whether the above result can be extended to graphs with unbounded degrees in a meaningful way. Recently, planar graphs with unbounded degrees have been attracting a lot of interest, in particular due to research on coarse geometry [10], random walks [4, 1, 5, 15] and random planar maps related to Liouville quantum gravity [2, 3, 9, 13, 16, 19, 22]. Motivated by this, our main result extends the aforementioned result of Benjamini & Schramm to unbounded degree graphs by replacing the transience condition with a stronger one, which we call *roundabout-transience* and explain below:

Theorem 1.1. *Let G be a locally finite roundabout-transient planar map. Then G admits a non-constant Dirichlet harmonic function.*

A *planar map* G, also called a *plane graph*, is a graph endowed with an embedding in the plane. The *roundabout graph* G° is obtained from G by replacing each vertex v with a cycle v° in such a way that the edges incident with v are incident with distinct vertices of v° (of degree 3), preserving their cyclic ordering; see also Section 4. We say that G is *roundabout-transient* if G° is transient[1]. In Section 4 we relate G° with circle packings of G.

Example 1.2. The aforementioned example of a 1-way infinite path with the n-th edge replaced by 2^n edges, is transient, but not roundabout transient. Indeed, each roundabout v° contains a cut consisting of just two edges separating the root from infinity. Thus the effective resistance to infinity is infinite in the roundabout graph, and Lyons' criterion (Theorem 2.3) implies recurrence.

We also provide a further way to strengthen the transience condition so as to guarantee Dirichlet harmonic functions. The idea is to require that there is a flow f from some vertex having not only finite Dirichlet energy, as required by Lyons' criterion, but also a finite norm in a different Hilbert space, obtained by giving weights to the edges depending on the degrees of their end-vertices. This is made precise in the following corollary of Theorem 1.1.

Corollary 1.3. *Let G be a locally finite planar graph G such that there is a flow f from some vertex v such that*

$$\sum_{vw \in E(G)} [deg(v)^2 + deg(w)^2] f(vw)^2 < \infty.$$

[1] The authors coined this term in Warwick, UK, where there are many roundabouts.

Then G admits a non-constant Dirichlet harmonic function.

As shown in Section 8, the order of magnitude of the weights here is best-possible. Hence Corollary 1.3 is tight, which indicates a way in which Theorem 1.1 is tight too.

Our work was partly motivated by a problem from [15], asking whether every simple planar graph with the Liouville property is (vertex-)*amenable*[2], by which we mean that for every $\epsilon > 0$ there is a finite set S of vertices of G such that less than $\epsilon|S|$ vertices outside S have a neighbour in S. As we show in Section 8,

Theorem 1.4. *Every locally finite non-amenable planar map is roundabout-transient.*

Combining this with Theorem 1.1 yields a positive answer to the aforementioned problem, and much more. This strengthens a result of Northschield [23], stating that every bounded degree non-amenable planar graph admits non-constant bounded harmonic functions, in two ways: it relaxes the bounded degree condition, and provides Dirichlet rather than bounded harmonic functions.

Benjamini [10] constructed a bounded degree non-amenable graph with the Liouville property. The last result shows that such a graph cannot be planar even if we drop the bounded degree assumption.

We think of Theorems 1.1 and 1.4 as indications that the notion of roundabout-transience is satisfied in many cases, and has strong implications. We expect it to find further applications. For example, we expect that the results of [24, Section 2] generalise from bounded-degree non-amenable planar maps to roundabout-transient ones. Moreover, one could try to extend the main results of [15] and [4], which identify the Poisson boundary of planar graphs with the boundary of the square tiling, and the circle packing respectively, from the bounded-degree transient case to the roundabout transient case, as we did in this paper for the result of Benjamini & Schramm on Dirichlet harmonic functions. Finally, perhaps the most interesting problem of this form is the following:

Problem 1.5. *Let G be the 1-skeleton of a triangulation of an open disc in $\mathbb{R}^2$. Is it true that G admits a circle packing in the unit disc if and only if it has a roundabout-transient subgraph?*

[2]For bounded degree graphs, vertex-non-amenability and the related notion of edge-non-amenability agree. For graphs with unbounded degrees like ours this is no longer the case, and we always mean vertex-non-amenable when writing non-amenable.

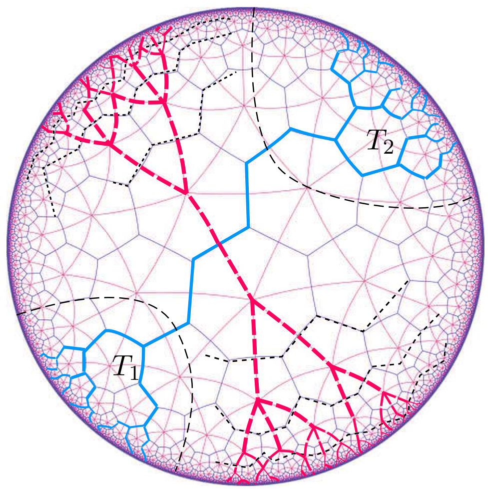

Figure 1: The two subgraphs T_1, T_2 delimited by the dashed curves are transient because of the blue flow. The dual of the black flow (dashed) witnesses the fact that the effective conductance between T_1 and T_2 is finite because it has finite energy.

If true, this would extend a well-known theorem of He & Schramm [18], stating that if G is recurrent, then it admits a circle packing whose carrier is the whole plane, and if it is transient and has bounded degrees, then it admits a circle packing in the unit disc. (It is known that every such graph admits a circle packing in either the whole plane or the unit disc, but not in both [17, 18, 26].) The reason why we do not conjecture that G admits a circle packing in the unit disc if and only if it is roundabout-transient itself in Problem 1.5, is that given any circle packing in the unit disc, it is always possible to insert enough new discs to make the contacts graph roundabout-recurrent. We leave this as an exercise for the interested reader.

We now give an overview of the proof of Theorem 1.1. As shown in [12], a graph admits non-constant Dirichlet harmonic functions if and only if it has two disjoint transient subgraphs T_1, T_2 such that the effective conductance between T_1 and T_2 is finite; see Theorem 3.1. To show that our graphs satisfy this condition, we start with a flow provided by Lyons' criterion. This flow lives in an auxiliary graph which for the purposes of this illustration can be thought of as a superimposition of G and its dual. We split this flow into four sub-flows, supported in disjoint regions of the plane, using the square tiling techniques of [15]. We use two of these subflows to obtain T_1, T_2, and we apply a duality argument to the other two to show that the effective conductance between T_1 and T_2 is finite; see Figure 1 and

Lemma 5.1. The latter step can be thought of as a variation of the idea that the effective resistance from the top to the bottom of a rectangle equals the effective conductance from left to right, with the latter two subflows showing finiteness of the top-to-bottom effective resistance.

2 Preliminaries

2.1 Graphs

We follow the terminology of [14] for graph-theoretic terms unless otherwise stated. All graphs in this paper are *directed*: a *graph*, G is a pair (V, E) where V is the set of vertices of G, and E is a set of directed pairs of elements of V, called the (directed) *edges* of G.

All our graphs are *simple*: they have no loops or parallel edges. (In the few occasions where we contract edges, one can subdivide any resulting parallel edges or loops to stay within the class of simple graphs.) A graph is *locally finite* if all its vertices have finite *degree*, where the degree of a vertex is the number of incident edges. Most graphs in this paper are locally finite. A locally finite graph G is *1-ended* if for every finite vertex S, the graph $G - S$ (obtained from G by deleting the vertices in S and their incident edges) has only one infinite component. Given a vertex set X, by $E(X)$ we denote those edges with both endvertices in X.

A *cut* of a graph G is the set of edges between a set of vertices $U \subseteq V(G)$ and its complement $V(G) \setminus U$.

2.2 Plane graphs

A graph is *planar*, if it admits an embedding in the plane $\mathbb{R}^2$. A *plane graph* is a (planar) graph endowed with a fixed embedding in the plane. We will be using the notion of the dual of a plane graph in the standard sense, but we adapt it to our directed graphs so that the directions of the edges of the primal determine the directions of the edges of the dual as follows. The *dual* of a plane (directed) graph $G = (V, E)$ is the graph $G^* = (F, E^*)$ whose vertex set is the set F of faces of G, having an edge e^* from a face v to a face w whenever G has an edge e incident with both v and w such that v lies on the right of e as we move along the direction of e. Note that by drawing the vertices of G^* inside the corresponding faces of G we can obtain an embedding in $\mathbb{R}^2$ such that $G^{**} = G$. To simplify notation we will, with a slight abuse, suppress the bijection $\cdot^*$ between the edge sets E, E^* of two dual plane graphs and pretend that $E = E^*$.

We will be using the following simple fact about plane dual graphs.

Lemma 2.1 ([14, Proposition 4.6.1])**.** *Let G and G^* be dual plane graphs, and suppose they are both locally finite. Then every minimal cut b of G forms a cycle C in G^* such that one of the components of $G-b$ lies in the interior of C and the other in its exterior*[3].

2.3 Electrical currents

Given a graph $G=(V,E)$ and a function $i:E\to\mathbb{R}$, the divergence $i^*(x)$ of i at a vertex x is the net flow out of x, that is, $i^*(x) := \sum_{xy\in E} i(xy) - \sum_{zx\in E} i(zx)$. We say that i satisfies *Kirchhoff's node law* at x if $i^*(x)=0$.

A *divergence free flow* is a function $i:E\to\mathbb{R}$ satisfying *Kirchhoff's node law* at every vertex. In an infinite graph it is possible for i to satisfy Kirchhoff's node law at all vertices except a single vertex o, at which we have $i^*(o)\neq 0$. In this case i is called a *flow from o* (to infinity). The *intensity* of i is the divergence $i^*(o)$. For a finite vertex-set A, we say that i is a *flow from A* if $i^*(x)>0$ for every $x\in A$. The *support* $supp(i)$ of i is the edge set $\{e\in E \mid i(e)\neq 0\}$.

A *potential* on G is a function $u:V\to\mathbb{R}$. The difference operator ∂ turns each potential $u:V\to\mathbb{R}$ into a function $\partial u:E\to\mathbb{R}$ by letting $\partial u(xy) := u(x)-u(y)$. If ∂u satisfies Kirchhoff's node law, then u satisfies the discrete Laplace equation:[4].

$$u(x) = \frac{\sum_{y\in E(x)} u(y)}{deg(x)}, \tag{1}$$

where $deg(x)$ denotes the degree of x. If u satisfies (1), then we say that u is *harmonic* at the vertex x. Note that the above implication can be reversed to yield that if a potential u is harmonic, then ∂u satisfies Kirchhoff's node law.

A potential $u:V\to\mathbb{R}$ is *harmonic* if it is harmonic at every vertex $x\in V$. The (Dirichlet) *energy* of a function $i:E\to\mathbb{R}$ is defined by

$$\mathcal{E}(i) := \sum_{e\in E} i^2(e).$$

The *energy of a potential* u is the energy of ∂u; in formulas: $\mathcal{E}(u) := \sum_{xy\in E}(u(x)-u(y))^2$. A harmonic function with finite Dirichlet energy is called a *Dirichlet harmonic function*.

[3] The statement that the components of $G-b$ lie in distinct sides of C is given in the proof of [14, Proposition 4.6.1] rather than in its statement.

[4] This can be seen by solving the equation $\sum_{y\in E(x)}(u(x)-u(y))=0$ for $u(x)$.

A *walk* in a graph G is a sequence $\{v_0, e_1, \dots e_k, v_k\}$ alternating between vertices and incident edges, starting and ending with a vertex. A walk is *closed* if its starting vertex v_0 is equal to its ending vertex v_k. Given a function $i : E \to \mathbb{R}$ and a closed walk $W = \{v_0, e_1, \dots e_k, v_k\}$, we define $curl_i(W) := \sum_{j \leq k} (-1)^{\delta_j} i(e_j)$, where $\delta_j = 1$ if W traverses e_j against its direction, and $\delta_j = 0$ otherwise. We say that i satisfies *Kirchhoff's* cycle law if $curl_i(W) = 0$ for every closed walk W in G (equivalently, if $curl_i(W) = 0$ for every closed walk W visiting no vertex —other than its starting vertex— more than once). It is not hard to check that

Observation 2.2. *A flow i satisfies Kirchhoff's cycle law if and only if there is a potential u with $i = \partial u$.*

2.4 Random walks

All random walks in this paper are simple and take place in discrete time, that is, if the random walker is at a vertex v of our graph G at time n, then at time $n + 1$ it is at a neighbour of v chosen uniformly at random. The starting vertex of our random walk will always be deterministic, and usually denoted by o.

A connected graph G is *transient* if random walk on G almost surely visits any fixed vertex finitely often. If G is not transient then it is *recurrent*. The following classical result of T. Lyons characterises transience in terms of flows.

Theorem 2.3 ([21], see also [20])**.** *A connected locally finite graph G is transient if and only if for some (and hence for every) vertex $o \in V(G)$, there is a flow from o in G with finite energy.*

Given a transient graph G and a vertex o, we can define a flow $i = i(o)$ from o as follows. For every vertex $v \in V$, let $h(v)$ be the probability $p_v(o)$ that random walk from v will ever reach o. In particular, $h(o) = 1$. By construction the potential h is harmonic at every $v \neq o$. Let $i = \partial h$. By our discussion in Section 2.3, i is a flow from o, and we call it the *random walk flow* from o.

3 Dirichlet harmonic functions

In this section we explain some of the tools we use in our proofs. The following results characterise the locally finite graphs admitting non-constant Dirichlet harmonic functions. We write $\mathcal{O}_{HD}$ for the class of graphs on which all Dirichlet harmonic functions are constant.

Theorem 3.1 ([12])**.** *A locally finite graph G admits a non-constant Dirichlet harmonic function if and only if there are two transient, vertex-disjoint, subgraphs A, B of G such that there is a potential of finite energy which is constant on each of A and B but not on $A \cup B$.*

Observation 3.2. *By adding a finite path to the subgraph A in Theorem 3.1 if necessary, and adapting the values of the potential on that path, we may assume that in the statement of Theorem 3.1 we moreover have an edge joining a vertex of A to a vertex of B.*

The following is a variant of Theorem 3.1 that is more convenient for our purposes in this paper.

Corollary 3.3. *A locally finite graph G admits a non-constant Dirichlet harmonic function if and only if it admits a divergence free flow f and a potential ρ, both of finite energy, such that the supports of f and $\partial\rho$ intersect in precisely one edge.*

Proof. To prove the forward implication, assuming that G is not in $\mathcal{O}_{HD}$, Theorem 3.1 and Observation 3.2, yield transient vertex-disjoint subgraphs A, B, connected by an edge e, as well as a potential ρ of finite energy which is constant on each of A, B but takes different values on them. Using the transience of A and B and Theorem 2.3 it is straightforward to construct a divergence free flow f of finite energy that is supported on the edges of $A \cup B$ and the edge e, with $f(e) \neq 0$. The supports of f and $\partial\rho$ then intersect only in the edge e as desired.

The backward implication can be shown using the methods of the proof of Theorem 3.1 in [12][5]. Here we take a different route; we will give a new functional analytic proof.

We consider the (real) Hilbert space H of functions from $E(G)$ to $\mathbb{R}$ with finite Dirichlet energy; our scalar product is defined by
$\langle f \mid g \rangle := \sum_{e \in E(G)} f(e)g(e)$.

The *cycle space* C of G is the subspace of H generated by the cycles of G; that is, for each cycle C_i of G, we let f_i be a non-zero divergence free flow supported on the edges of C_i (f_i is determined by C_i up to a multiplicative constant that does not matter), and let C be the subspace of H generated by all the f_i.

The *cut space* D of G is the subspace of H generated by the finite cuts of G: for every finite cut D_i of G, with corresponding bipartition $V_1, V_2 \subset V$,

[5] More precisely, from the existence of f and ρ as in that theorem, one can construct transient subgraphs A and B as in Theorem 3.1.

pick any $c \in \mathbb{R}, c \neq 0$, and let d_i be the function (supported on D_i) satisfying $d_i(e) = c$ if e goes from V_1 to V_2 and $d_i(e) = -c$ if e goes from V_2 to V_1. Finally, let D be the subspace of H generated by all these d_i.

Note that C and D are orthogonal spaces, since each cycle crosses each cut an even number of times. Moreover, every divergence free flow lies in $D^\perp$: it is straightforward to check that $f \in D^\perp$ if and only if f satisfies Kirchhoff's node law at every vertex. Furthermore, $C^\perp$ coincides with the space $\{\partial u \mid u \text{ is a potential}\}$. Therefore, to show that G admits a non-constant Dirichlet harmonic function, it suffices to show that $D^\perp \cap C^\perp$ is non-trivial, as all functions in $D^\perp$ satisfy Kirchhoff's node law, and so their corresponding potentials are harmonic by the discussion in Section 2.3.

Let us apply these observations to the functions f and ρ of the statement. The assumption that f and $\partial\rho$ intersect in precisely one edge implies that $\langle f \mid \partial\rho \rangle \neq 0$.

As D is orthogonal to C, we have $C \subseteq D^\perp$, and so we can decompose $D^\perp$ as $D^\perp = C + (D^\perp \cap C^\perp)$. Thus we can write our $f \in D^\perp$ as $f_1 + f_2$ with $f_1 \in D^\perp \cap C$ and $f_2 \in D^\perp \cap C^\perp$. Since $\partial\rho \in C^\perp$, we have $\langle f_1 \mid \partial\rho \rangle = 0$, and since $\langle f \mid \partial\rho \rangle \neq 0$ we must have $\langle f_2 \mid \partial\rho \rangle \neq 0$. In particular, $f_2 \neq 0$ and so we have proved our claim that $D^\perp \cap C^\perp \ni f_2$ is non-trivial. □

Corollary 3.4 ([27])**.** *Let G be a locally finite graph with a finite cut b such that $G - b$ has two transient components. Then G is not in $\mathcal{O}_{HD}$.*

Proof. We apply Theorem 3.1, with ρ being e.g. the potential defined by $\rho(x) = i$ for every x in C_i, where C_i is the ith component of $G - b$ in some enumeration of those components. □

Definition 3.5. Given a locally finite graph G, and a subgraph $H \subseteq G$, we will say that a function $f : E(G) \to \mathbb{R}$ *witnesses that H is transient*, if the restriction f_H of f to $E(H)$ is a flow from some finite vertex set (to infinity) with finite energy.

As we can easily modify f_H at finitely many edges to turn it into flow from a single vertex (to infinity), such an f_H implies that H is transient by Theorem 2.3.

Observation 3.6. *Let G and G^* be locally finite dual plane graphs. Let f be a divergence free flow in G with finite energy. Then one of the following is true.*

A) The function f satisfies Kirchhoff's cycle law in G^*;

B) there is a finite cut c of G such that f witnesses that at least two components of $G - c$ are transient.

Proof. Suppressing the bijection $\cdot^*$ between the directed edges of G and G^*, the function f can be thought of as a function on $E(G^*)$. If f fails to satisfy (A), then there is a finite cycle C of G^* at which f violates Kirchhoff's cycle law. Since G and G^* are dual, the edges of C form a cut C^* of G, separating it into two subgraphs U, W. Moreover, our assumption on C means that the net flow of f from U to W is non-zero. Thus f_U is a flow from a finite set (namely, from those vertices of U incident with an edge in C^*) witnessing that U is transient. Similarly, f_W witnesses that W is transient too. □

Remark 3.7. For finite plane dual graphs G and G^*, a function f satisfies Kirchhoff's cycle law in the graph G if and only if it satisfies Kirchhoff's node law in the dual graph G^*. Observation 3.6 could be understood as an extension of this fact.

4 Roundabout-transience

The *roundabout graph* G° of a locally finite plane graph G is obtained from G by replacing each vertex v by a cycle (*roundabout*) of length equal to the degree of v so that every vertex gets degree 3; formally, the vertex set of G° is the set of pairs (v, e) where e is an edge and v is an endvertex of e. The embedding of G defines the (clockwise, say) cyclic order C_v on the set of edges incident with the vertex v. The edges of G° are of two types; for each edge $e = \overrightarrow{vw} \in E(G)$, we have an edge in G° from (v, e) to (w, e). For any two consecutive edges e and f in the cyclic order C_v, we have an edge in G° from (v, e) to (v, f).

The roundabout graph G° has a canonical embedding in the plane, namely, the one that induces the embedding of G when we contract each roundabout into a single vertex.

With a slight abuse of notation, we will treat $E(G)$ as a subset of $E(G^\circ)$, with the understanding that $e = \overrightarrow{vw} \in E(G)$ is identified with $\overrightarrow{(v, e)(w, e)} \in E(G^\circ)$.

We say that a graph G is *roundabout-transient* if G° is transient.

Observation 4.1. *Every cut of G is a cut of G°.*

Conversely, every cut b of G° with $b \subseteq E(G)$ is also a cut of G. □

Remark 4.2. The structure of G° depends on the chosen embeddings of G. Here, we construct a planar graph G that has both a transient and a recurrent roundabout graph (corresponding to different embeddings).

Let G be the graph obtained from the infinite binary tree[6] T_2 by attaching 2^n leaves at each vertex at distance n from a fixed root of T_2. Let G_1 be the plane graph obtained by embedding G in the plane in such a way that all leaves attached to v are embedded consecutively for every $v \in V(T_2)$. It is straightforward to check that G_1° is transient: by deleting all leaves and their incident vertices we obtain a subgraph of G_1° quasi-isometric to T_2. Let G_2 be the plane graph obtained by embedding G in the plane in such a way that the leaves attached at each v are separated into two equal subsets by the edges of T_2. It is not hard to check that G_2° is recurrent: the leaves now have the effect of introducing long subdivisions to T_2.

To summarise, roundabout-transience is a property of plane graphs and not of planar graphs.

Lemma 4.3. *If G° is transient, then so is G.*

Proof. Since G° is transient, it admits a flow f of finite energy from some vertex $o \in V(G^\circ)$ by Lyons' criterion Theorem 2.3. We will show that f induces a flow of finite energy in G.

For a vertex $v \in V(G^\circ)$, let us denote by v° the set of vertices lying in the same roundabout as v. Note that f satisfies Kirchhoff's node law at every vertex-set v° except o°. Therefore, the restriction f' of f to the edges of G satisfies Kirchhoff's node law at every vertex of G except the vertex o' that gave rise to o°. In other words, f' is a flow from o'. Its energy is bounded from above by that of f, and so G is transient by Theorem 2.3. □

In the following we will often use the notation $G^{*\circ}$, by which we mean the roundabout graph $(G^*)^\circ$ of the dual G^* of the plane graph G.

The *plane line graph* $G^\diamond$ of a plane graph G is the plane graph obtained from G° by contracting all (non-roundabout) edges of G. Another way to define $G^\diamond$, explaining its name, is by letting the vertex set of $G^\diamond$ be the set of midpoints of edges of G and joining two such points with an arc whenever the corresponding edges are incident with a common vertex v of G and lie in the boundary of a common face of v. It is clear from this definition that

$$G^\diamond = (G^*)^\diamond =: G^{*\diamond}. \tag{2}$$

A third equivalent definition of $G^\diamond$ can be given by considering a circle packing P of G, letting $V(G^\diamond)$ be the set of intersection points of circles of

[6] The *binary tree* is the unique infinite tree in which every vertex has degree three.

P, and letting the arcs in P between these points be the edges of $G^\diamond$. A fourth definition of $G^\diamond$ is as the dual of the bipartite graph G', with $V(G')$ consisting of the vertices and faces of G, and $E(G')$ joining each vertex of G to each of its incident faces.

It is easy to see that $G^\diamond$ is quasi-isometric (in fact Bilipschitz-equivalent) to G°. Since both graphs have bounded degrees, Theorem 2.3 easily implies the following.

Lemma 4.4. *Let G be a locally finite plane graph. Then G° is transient if and only if $G^\diamond$ is.* □

Lemma 4.4, combined with the fact that $G^\diamond = G^{*\diamond}$ (2), yields that if G° is transient, then so is $G^{*\circ}$. Another way to state this is:

G is roundabout-transient if and only if G^* is. (3)

Combining this with Lemma 4.3, we obtain

Corollary 4.5. *if G° is transient, then so is G^*.*

5 Square tilings and the two crossing flows

5.1 Square tiling basics

In this section we use the theory of square tilings of (locally finite) transient plane graphs in order to find the special flows in our roundabout-transient G mentioned in the introduction. These square tilings were introduced in [6], and generalise a classical construction of Brooks et. al. [11] from finite plane graphs to infinite transient ones.

Let $\mathcal{C}$ denote the cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0, 1]$, or more generally, a cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0, a]$ for some real $a > 0$ (which will turn out to coincide with the effective resistance from a vertex o to infinity). A *square tiling* of a plane graph G is a mapping τ assigning to each edge e of G a square $\tau(e)$ contained in $\mathcal{C}$, where we allow $\tau(e)$ to be a 'trivial square' consisting of just a point (see Figure 2 for an example). A nice property of square tilings is that every vertex $x \in V$ can be associated with a horizontal line segment $\tau(x) \subset \mathcal{C}$ such that for every edge e incident with x, $\tau(e)$ is tangent to $\tau(x)$.

The construction of this τ is based on the random walk flow i from a root vertex o (as defined in Section 2.4): the side length of the square $\tau(e)$ is chosen to be $|i(e)|$, and the placement of that square inside $\mathcal{C}$ is decided by a coordinate system where potentials of vertices induced by the flow i are used as coordinates. For example, the top circle of the cylinder $\mathcal{C}$ is the 'line segment' $\tau(o)$ corresponding to o, because o has the highest potential. All other vertices and edges accumulate towards the base of $\mathcal{C}$, because their potentials (which equal the probability for random walk from those vertices to return to o, normalised by the height of $\mathcal{C}$) converge to 0; see [15] for details.

We let $w(\tau(e))$ denote the width of the square $\tau(e)$. Our square tilings always have the following properties which we will use below:

I) Two of the sides of $\tau(e)$ are always parallel to the boundary circles of $\mathcal{C}$;

II) $w(\tau(e)) = |i(e)|$ for every $e \in E$, where i denotes the random walk flow out of o;

III) the interiors of any two such squares $\tau(e), \tau(f)$ are disjoint;

IV) every point of $\mathcal{C}$ lies in $\tau(e)$ for some $e \in E$;

V) every vertex x can be associated with a horizontal line segment[7] $\tau(x) \subset \mathcal{C}$ so that for every edge e incident with x, the square $\tau(e)$ is tangent to $\tau(x)$, and every point of $\tau(x)$ is in $\tau(f)$ for some edge f incident with x, and

VI) every face F can be associated with a vertical line segment $\tau(F) \subset \mathcal{C}$ so that for every edge e in the boundary of F, the square $\tau(e)$ is tangent to $\tau(F)$.

It was shown in [6] that a plane graph G admits a square tiling exactly when G is *uniquely absorbing*. We say that G is *uniquely absorbing*, if for every finite subgraph G_0 there is exactly one connected component D of $\mathbb{R}^2 \setminus G_0$ which is *absorbing*, that is, random walk on G visits $G \setminus D$ only finitely many times with positive probability (in particular, the subgraph of G embedded in D is transient, hence so is G).

5.2 Cutting the random walk flow along meridians

A *meridian* of $\mathcal{C}$ is a vertical line of the form $\{x\} \times \{0,1\} \subset \mathcal{C}$ for some $x \in \mathbb{R}/\mathbb{Z}$. Meridians are important, as they will allow us to 'dissect' sub-flows of the random walk flow i. To make this precise, given a vertex $x \in V(G)$, we let $|x|$ denote the vertical 'strip' of the cylinder $\mathcal{C}$ whose horizontal span coincides with that of the line segment $\tau(x)$ as described in (V): we let

[7] $\tau(x)$ might be a full horizontal circle of $\mathcal{C}$. This is always the case for $x = o$.

$|x| := I \times \{0, a\} \subset \mathcal{C}$, where I is the interval of coordinates appearing in $\tau(x)$. Then $\tau(x)$ separates $|x|$ into two rectangles, and we denote the bottom one (that is, the one not meeting $\tau(o)$) by $\lceil x \rceil$.

Next, we associate to this x a flow $\check{x}$ from x that 'lives in $\lceil x \rceil$'. Let us assume that each edge $e = vw$ of G is directed 'downwards', that is, the height coordinate of $\tau(v)$ is higher than that of $\tau(w)$; we can make this assumption without loss of generality as we can always change the direction of an edge simultaneously with the sign of its flow. To define the flow $\check{x}$, for every $e \in E(G)$, let $\check{x}(e) := w(\tau(e) \cap \lceil x \rceil)$ be the width of the rectangle $\tau(e) \cap \lceil x \rceil \subset \mathcal{C}$ corresponding to e. (Thus if $\tau(e)$ is contained in $\lceil x \rceil$, then $\check{x}(e) = i(e)$ by (II), where i is again the random walk flow from o, and if $\lceil x \rceil$ dissects $\tau(e)$, then $\check{x}(e) < i(e)$.) A basic property of meridians (already observed in [15, Lemma 6.6]), is that $\check{x}$ is indeed a flow from x: to see this, let $v \neq x$ be any vertex such that $\tau(v)$ intersects $\lceil x \rceil$, and note that $\check{x}$ brings flow into v using the edges whose squares are tangent to $\tau(v)$ from above, and it removes flow into v using the edges whose squares are tangent to $\tau(v)$ from below, and the total intensity of both these contributions equals the length of the intersection of $\tau(v)$ with $\lceil x \rceil$.

More generally, if M, M' are two meridians intersecting $\tau(x)$, we let $\lceil MxM' \rceil$ denote the rectangle of $\mathcal{C}$ bounded by $M, \tau(x), M'$ and the bottom circle of $\mathcal{C}$, and define the flow from x that *lives in* $\lceil MxM' \rceil$ similarly to $\check{x}$, except that we replace the rectangle $\lceil x \rceil$ with $\lceil MxM' \rceil$ in the above definition.

5.3 The basic lemma

The following lemma makes use of a square tiling to perform a certain 'surgery' on the random walk flow i on the plane line graph $G^\diamond$ of a roundabout transient graph G. By recombining pieces of i appropriately, we induce flows on G° and $G^{*\circ}$ (or rather, on finite modifications of those graphs) that we will later use to make the intuition of Figure 1 precise.

Every flow i on $G^\diamond$ induces a flow $i_\circ$ on G°, called the *lift* of i to G°, as follows. For every edge $e \in E(G^\diamond)$, we recall that e is also an edge of G°, and just set $i_\circ(e) = i(e)$. For every other edge e of G°, we let $i_\circ(e)$ be the unique value such forces $i_\circ$ to satisfy Kirchhoff's node law at both endvertices u, v of e. Such a value exists because i satisfies Kirchhoff's node law, and so the total divergence of u, v in $i_\circ$ is 0 for any value of $i_\circ(e)$.

Lemma 5.1. *Let G and G^* be locally finite dual plane graphs. If G° is transient, then for some graph H obtained from G by contracting a finite* *connected subgraph into a vertex, there are divergence free flows f and h of finite energy in H° and $H^{*\circ}$ respectively, the supports of which intersect in a single edge (of $E(H) = E(H^*)$).*

The proof of this is a bit technical, but the main idea is quite simple. Let us assume that $H = G$ for a moment to explain the intuition. The interesting case is where G° is uniquely absorbing, in which case we can make use of the square tiling (of $G^\diamond$ rather than G° for technical reasons). In this case, we use certain pairs of meridians to 'dissect' four sub-flows f_j, from four distinct vertices x_j to infinity, of the random walk flow on $G^\diamond$ that live in four disjoint narrow rectangles of the tiling cylinder $\mathcal{C}$ of $G^\diamond$ using the definitions of Section 5.2. Combining these flows in pairs using two finite flows, one from x_1 to x_3, and one from x_2 to x_4, we obtain two divergence free flows f', h' in $G^\diamond$ that 'cross' in a manner corroborating the intuition of Figure 1. It is then straightforward to lift f', h' to the desired flows f, h in the two roundabout graphs using the above definition.

The statement of Lemma 5.1 may be a bit confusing, as it involves several graphs with shared edges. Our choice to work with $G^\diamond$ may seem to be making matters worse at first sight, as it introduces one more graph. However, it makes life easier: rather than having to work with several graphs simultaneously, all non-trivial parts of the following proof deal with just one graph, $G^\diamond$. The nice aspect of $G^\diamond$ is that it provides a concise representation of the graphs G, G^*, G° and $G^{*\circ}$. The important property to remember is that the vertex set of $G^\diamond$ is the (common) edge-set of G and G^*, which is also the intersection of $E(G^\circ)$ and $E(G^{*\circ})$. Since the objective of Lemma 5.1 is a pair of divergence free flows in G°, $G^{*\circ}$ with a single common edge, this boils down to finding two divergence free flows in $G^\diamond$ that cross at a single vertex. For technical reasons, it is a bit easier to find a pair of flows with finitely many crossings, and therefore we introduce the auxiliary graph H: after modifying a finite part of the graph where all crossings take place, it is easier to end up with a single crossing.

Proof of Lemma 5.1. We distinguish two cases, according to whether $G^\diamond$ is uniquely absorbing or not.

If $G^\diamond$ is uniquely absorbing, then [6] provides a square tiling of $G^\diamond$ on a cylinder $\mathcal{C}$ as described above, with o being an arbitrary vertex of $G^\diamond$.

Our plan is to find four vertices $x_1, \ldots, x_4$ far enough from each other on $\mathcal{C}$ and flows f_j from those vertices that live in appropriate disjoint rectangles, and combine these flows pairwise to obtain f', h'. More precisely, we claim that we can choose four vertices $x_j, 1 \leq i \leq 4$ in $G^\diamond$, a flow f_j from each

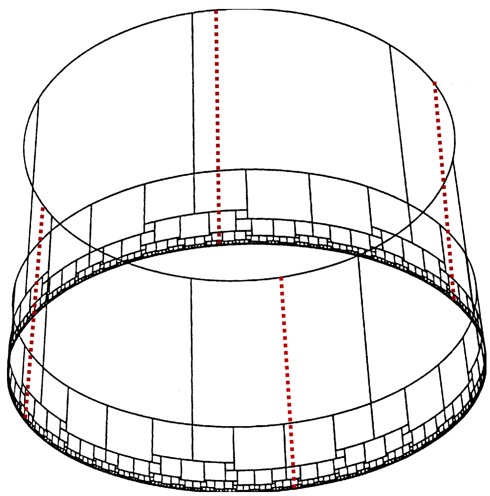

Figure 2: An example of a square tiling, with the four meridians M_j of Lemma 5.1 in dotted lines.

x_j, and a path P_j from x_j to o, so that these objects satisfy the following properties, which can be summarised by saying that these objects avoid to meet a common roundabout of $G^\diamond$ whenever possible[8].

A) $\mathrm{supp}(f_k) \cap \mathrm{supp}(f_j) = \emptyset$ for $k \neq j$; even stronger, no roundabout of G° meets both $\mathrm{supp}(f_k)$ and $\mathrm{supp}(f_j)$;

B) for every $j \leq 4$ and every edge e of P_j, no edge of the roundabout of G° containing e is in the support of any $f_k, 1 \leq k \leq 4$, and

C) the roundabout of G° containing the first edge of P_k does not contain x_j and does not contain any edge of P_j for $j \neq k$.

Before proving that such a choice is possible, let us first see how it helps us construct the desired divergence free flows f, h.

Let X be the set of vertices v of G such that the roundabout v° in $G^\diamond$ contains an edge of P_j but does not contain the first edge of P_j. By construction, X spans a connected subgraph of G, since all P_j contain o. Let H be the graph obtained from G by contracting X into a single vertex x.

[8]Recall that $G^\diamond$ is obtained from G° by contracting all edges outside roundabouts. Whenever we talk about a *roundabout of* $G^\diamond$ we will mean a roundabout of G° considered as a subgraph of $G^\diamond$.

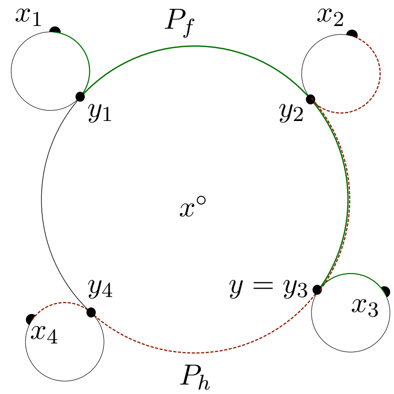

Figure 3: The roundabouts O_1, O_2, O_3, O_4 and x°, along with the x_1–x_3 path P_f (shown in green, if colour is shown) and the x_2–x_4 path P_h (dashed, red) used in the definition of f', h', respectively.

It is straightforward to see that $H^\diamond$ can be obtained from $G^\diamond$ by replacing all roundabouts corresponding to vertices in X by the single roundabout x°. The desired flows f, h will be obtained as lifts —as defined before the assertion of Lemma 5.1— of auxiliary flows f', h' in $H^\diamond$ constructed as follows. By the construction of H, the first edge of each P_j lies in a roundabout O_j that shares a vertex y_j with x°, and $O_j \neq O_k$ for $j \neq k$. In particular, x_j lies on O_j too; see Figure 3. Note that O_j might have several vertices in common with x°, because H was obtained from G by a contraction that may have introduced parallel edges. In this case, we choose y_j so that O_j contains an x_j–y_j path Q_j that only meets x° at y_j.

Assume without loss of generality that y_1, y_2, y_3, y_4 appear in that order as we move around x° clockwise. We will construct a divergence free flow f' as a linear combination of f_1, f_3, and a constant flow from x_1 to x_3 along a path P_f contained in $O_1 \cup x^\circ \cup O_3$. We choose this P_f to be a concatenation of Q_1, one of the two y_1–y_3 paths contained in x°, and Q_3. To make the definition of f' precise, suppose the intensity[9] of f_1 is $\beta \in \mathbb{R}_+$, and the intensity of f_3 is $\gamma \in \mathbb{R}_+$. For each edge $e \in supp(f_1)$, we set $f'(e) = f_1(e)/\beta$. For each edge $e \in supp(f_3)$, we set $f'(e) = -f_3(e)/\gamma$. Finally, for each edge e of P_f, we set $f'(e) = 1$ if the direction of e agrees with that of P_f (which is from x_1 to x_3), and $f'(e) = -1$ otherwise. It is straightforward to check that f' satisfies Kirchhoff's node law.

Similarly, we construct the flow h' as a linear combination of f_2, f_4, and a constant flow from x_2 to x_4 along a path P_h contained in $O_2 \cup x^\circ \cup O_4$,

[9] Recall that the intensity of f_j is the divergence $f_j^*(x_j)$.

obtained by concatenating Q_2 and Q_4 with one of the two y_2–y_4 paths contained in x°. Finally, let f be the lift of f' to H° and let h be the lift of h' to $H^{*\circ}$.

We claim that these flows satisfy our requirement $|supp(f) \cap supp(h)| = 1$. To see this, we observe that there is a unique vertex y of x° at which P_f switches between two roundabouts of H° and simultaneously P_h switches between two roundabouts of $H^{*\circ}$[10]. Indeed, P_f stays within a roundabout of H° except precisely at the vertices y_1 and y_3, where it switches from O_1 to x° and from x° to O_3 respectively. Moreover, P_h contains exactly one vertex $y \in \{y_1, y_3\}$, and it contains two edges of x° incident with y, therefore it switches between the two roundabouts of $H^{*\circ}$ incident with y.

From this, it follows immediately from the definition of a lift that the unique edge (of $E(H) = E(H^*)$) in $supp(f) \cap supp(h)$ incident with x is the edge corresponding to y. Finally, no edge that is not incident with x can lie in $|supp(f) \cap supp(h)|$ by properties (A)–(C): these properties where designed exactly so us to prevent further intersections.

Thus, in the uniquely absorbing case, it only remains to prove that we can indeed choose vertices x_j, flows f_j, and paths P_j with properties (A), (B) and (C) above.

For this, recall that the length of the circumference of $\mathcal{C}$ is 1, and let $M_j, 0 \leq j < 4$ denote the meridian of $\mathcal{C}$ whose width coordinate is $j/4$. For each j, let $h_j \in (0, \frac{1}{16})$ be small enough that every roundabout of G° meeting M_j at a point whose height coordinate is less than h_j has width less than $1/8$, where the width of a roundabout O is defined to be the maximum width of a horizontal line segment contained in $\tau[O]$; such a choice is possible because $\tau[O]$ is two squares wide at each horizontal level by (VI) (where we use the fact that O bounds a face of $G^\diamond$), and a square that starts close to the bottom of $\mathcal{C}$ cannot be very wide. In addition, we choose h_j even smaller, if needed, to ensure that if x is a vertex such that $\tau(x)$ meets M_j below height h_j, then $w(\tau(x)) < 1/8$; this is possible because there are only finitely many edges e with $w(\tau(e))$ greater than any fixed constant since $\mathcal{C}$ has finite area, and $\tau(x)$ is at most three squares $\tau(e)$ wide by (V) and the fact that $G^\diamond$ is 4-regular.

Let $\lceil h_j M_j \rceil$ denote the sub-interval of M_j with height coordinates ranging between zero and h_j, and $\lfloor h_j M_j \rfloor$ the sub-interval of M_j with height coordinates ranging between h_j and 1.

[10]In the example of Figure 3, we have $y = y_3$. If we had lifted f' to $H^{*\circ}$ and h' to H° instead, then we would have had $y = y_2$. If we had chosen a P_f that uses the other y_1–y_3 path of x°, then we would have had $y = y_4$.

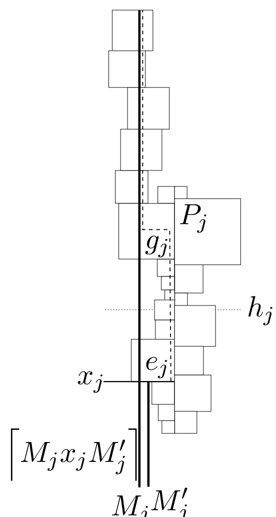

Figure 4: The choice of x_j, f_j and P_j.

It is proved in [7, Theorem 4.1 (v)] that almost every meridian with respect to Lebesgue measure meets only finitely many squares of the tiling lying above any fixed height. We may assume that our $M_j, 0 \leq j < 4$, all have this property, for otherwise we can achieve it by rotating $\mathcal{C}$. Therefore, for every $j < 4$, there is a lowest square $\tau(e_j)$ meeting $\lceil h_j M_j \rceil$ such that the roundabout O_j of G° containing the edge e_j also contains an edge g_j meeting $\lfloor h_j M_j \rfloor$ (Figure 4); this is true because $\lfloor h_j M_j \rfloor$ only meets finitely many squares of positive area, and so there are finitely many roundabouts to choose from. There is at least one to choose from: a roundabout whose image contains the point of M_j at height h_j.

Let x_j denote the endvertex of e_j whose height coordinate is lower, and note that $\tau(x_j)$ meets M_j. Let M'_j be a meridian meeting $\tau(e_j)$ (and in particular $\tau(x_j)$) close enough to M_j, but distinct from M_j, that the rectangle $\left\lceil M_j x_j M'_j \right\rceil$ bounded by M_j, x_j, M'_j and the bottom circle of $\mathcal{C}$, meets the τ image of no roundabout meeting $\lfloor h_j M_j \rfloor$; such a M'_j exists because, by the choice of e_j, O_j, no roundabout meeting $\lfloor h_j M_j \rfloor$ has an edge e meeting $\left\lceil M_j x_j M'_j \right\rceil$, or we would have chosen e instead of e_j. As we can choose M'_j as close to M_j as we wish, we may assume that $d(M_j, M'_j) < 1/16$, which will be useful later.

Let f_j be the flow from x_j that lives in $\left\lceil M_j x_j M'_j \right\rceil$, as defined at the beginning of Section 5.3. We claim that

If $e \in \operatorname{supp}(f_j)$, then $\tau(e)$ is contained in the open vertical strip of radius $1/8$ centered at M_j. (4)

Indeed, by the definition of f_j, if $e \in \operatorname{supp}(f_j)$, then $\tau(e)$ intersects the interior of $\left\lceil M_j x_j M_j' \right\rceil$. Then $\tau(e)$ cannot have a point at height higher that h_j, which we recall is less than $1/16$, because it would have to intersect the interior of $\tau(e_j)$ in that case, contradicting (III). Thus the height of $\tau(e)$ is at most $1/16$, and being a square, so is its width. Together with our assumption that $d(M_j, M_j') < 1/16$, this proves our claim.

Note that (4), combined with the choice of the M_j, immediately implies that $\operatorname{supp}(f_k) \cap \operatorname{supp}(f_j) = \emptyset$ for $k \neq j$; in fact, it even implies the stronger statement of (A), because by (4) if edges e, f lie in a common roundabout then $\tau(e), \tau(f)$ must meet a common meridian.

It remains to construct the paths P_j: we let P_j start with the x_j-g_j path in O_j containing e_j, and continue with the g_j-o path consisting of all the edges whose τ-image meets M_j above $\tau(f_j)$. Recall that there are only finitely many such squares as we remarked above. The fact that the edges whose τ-image meets M_j above $\tau(g_j)$ form a g_j-o path follows from (V) and the fact that $\tau(o)$ is the top circle of $\mathcal{C}$. In fact, by the above argument, we can even assume that M_j does not meet the boundary of any square $\tau(e)$, and so M_j uniquely determines that g_j-o path. Note that by construction,

every edge of P_j is in a roundabout O such that $\tau[O]$ meets M_j. (5)

To see that (B) is satisfied, recall that we chose h_j small enough that every roundabout of G° meeting M_j at a point whose height coordinate is less than h_j has width less than $1/8$, and P_j only uses roundabouts meeting M_j. Thus for $e \in E(P_j)$, $\tau(e)$ is contained in the vertical strip of radius $1/8$ centered at M_j. On the other hand, (4) says that the support of f_j is contained in the strip of radius $1/8$ centered at M_j, and so (B) follows from the fact that $d(M_k, M_j) \geq 1/4$.

Finally, we can prove (C) by a similar argument, now using the fact that $w(\tau(x_j)) < 1/8$ by the second part of our definition of h_j, and the fact that the roundabout containing the first edge e_j of P_j is contained in the strip of radius $1/8$ centered at M_j and every roundabout containing an edge of P_j meets M_j by (5).

Thus all three desired properties (A)–(C) are satisfied, and as discussed above this completes the case where $G^\diamond$ is uniquely absorbing.

Suppose now $G^\diamond$ is not uniquely absorbing. Then for some finite subgraph G_0 of $G^\diamond$, we have at least two absorbing components D_1, D_2 in $\mathbb{R}^2 \backslash G_0$. By elementary topological arguments, G_0 contains a cycle C such that both the interior I and the exterior O of C contain transient subgraphs of $G^\diamond$, namely one of its face boundaries.

If any of these subgraphs I, O is uniquely absorbing, then we can repeat the above arguments to that subgraph to obtain the two desired flows.

Hence it remains to consider the case where there is a cycle C_I in I and a cycle C_O in O that further separate each of I, O into two transient sides. In fact, we can iterate this argument as often as we like, to obtain many distinct transient subgraphs separated from any given cycle. Let us iterate it often enough to obtain four disjoint cycles $C_j, 1 \leq 4$, and inside each C_j a cycle D_j such that the interior of D_j is transient and no roundabout ot G° meets any two of these eight cycles.

We now apply Theorem 2.3 to each of the four interior sides of the D_j to obtain four flows of finite energy f_j from vertices x_j, such that the support of f_j is contained in D_j. We can then combine those flows pairwise in a way similar to the uniquely absorbing case to obtain the two desired flows f, h: we can let o be an arbitrary vertex outside all C_j, pick paths P_j from x_j to o, and again consider a graph H obtained from G by contracting the vertices corresponding to all roundabouts meeting the P_j except for the first one. We then construct f', h', and from them f, h, as indicated in Figure 3. The fact that $|\operatorname{supp}(f) \cap \operatorname{supp}(h)| = 1$ follows from the same graph-theoretic arguments about the structure of $G^\diamond$, for which we did not need the square tiling. $\square$

6 Harmonic functions on plane graphs

In this section, we use Theorem 3.1 to prove a new existence criterion for non-constant Dirichlet harmonic functions in planar graphs, Theorem 6.3 below, which is used in the proof of Theorem 1.1. Before proving Theorem 6.3, we prove the following which may be of independent interest, and can be thought of as a warm-up towards the harder Theorem 6.3. The reader will lose nothing by skipping directly to Theorem 6.3.

Theorem 6.1. *Let G and G^* be locally finite 1-ended dual plane graphs. Then the following are equivalent:*

A) $G \notin \mathcal{O}_{HD}$;

B) $G^* \notin \mathcal{O}_{HD}$;

C) there are divergence free flows f and h of finite energy in G and G^*, respectively, whose supports intersect in a single edge.

Proof. By symmetry, it suffices to show that (A) is equivalent to (C). For this, assume first that $G \notin \mathcal{O}_{HD}$. Then by Corollary 3.3 G admits a divergence free flow f and a potential ρ such that both f and $\partial\rho$ have finite energy and their supports intersect in a single edge. As $\partial\rho$ satisfies Kirchhoff's cycle law in G, when considered as a function on the dual G^* it satisfies Kirchhoff's node law at every vertex; that is, $\partial\rho$ is a divergence free flow of G^*. Hence f and $h := \partial\rho$ satisfy (C).

For the converse, suppose (C) holds. Consider h as a function on the edges of G. We are going to apply Observation 3.6 to G^* to deduce that h satisfies Kirchhoff's cycle law in G. Since G is one-ended, item (B) of Observation 3.6 cannot be satisfied, hence item (A) applies and says that h satisfies Kirchhoff's cycle law in G. Thus by Observation 2.2 there is a potential ρ in G with $\partial\rho = h$, and so by Corollary 3.3 the flow f and the potential ρ witness that $G \notin \mathcal{O}_{HD}$. □

Example 6.2. We give a simple example of a graph G such that neither the second nor the third condition imply the first in Theorem 6.1 if we omit the assumption that G and G^* are 1-ended. We first construct an auxiliary graph H from the disjoint union of a family of cycles $C_n, n \in \mathbb{N}$, where C_n has length 2^n, by gluing C_n and C_{n+1} together along an edge for each $n \geq 2$; we choose the two gluing edges in C_n so that they have distance $|C_n|/2 - 1$. We obtain the graph G by attaching two copies of H at distinct vertices of a triangle T. Clearly, the graph G is in $\mathcal{O}_{HD}$. But we claim that it satisfies the second and third condition.

To see this, we consider the embedding of G in the plane indicated in Figure 5. The dual G^* corresponding to this embedding is a 1-way infinite path with many parallel edges; in fact the removal of any vertex splits it into two transient subgraphs. Easily, G^* has a Dirichlet harmonic function (see e.g. [27, Theorem 4.20]). To see that the third condition is satisfied, we let f be a divergence free flow in G supported on the triangle T, and let h be a flow with infinite support in G^* that uses only one edge of T; the latter exists because the intersection of each side of T with G^* is transient.

The next result provides a strengthening of condition (C) of Theorem 6.1 which implies that $G \notin \mathcal{O}_{HD}$ even if G has more than one end.

Theorem 6.3. *Let G and G^* be locally finite dual plane graphs such that their roundabout graphs G° and $G^{*\circ}$ admit divergence free flows f and h*

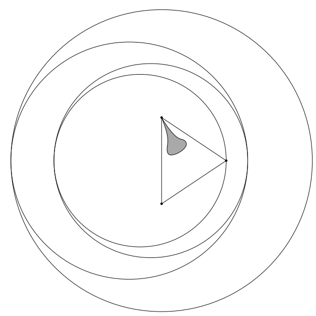

Figure 5: An embedding of the graph G in the plane. One copy of H is embedded on the outside of the triangle. The other copy is embedded in the grey region in an analogous way (here we embed the cycle C_{n+1} inside C_n).

respectively, both of finite energy, the supports of which intersect in a single edge (of $E(G) = E(G^*)$). Then $G \notin \mathcal{O}_{HD}$.

Proof. Since divergence free flows satisfy Kirchhoff's node law at finite vertex-sets, the restriction of the flow h of $G^{*\circ}$ to the edges of G^* is a divergence free flow in G^*. We denote that flow by h_{G^*}. We distinguish two cases.

Case 1: the flow h_{G^*} considered as a function on the edges of G satisfies Kirchhoff's cycle law in G.

Then $h_{G^*} = \partial\rho$ for some potential ρ on G by Observation 2.2. As above, the restriction of f to the edges of G is a divergence free flow f_G in that graph. Then f_G and the potential ρ of G witnesses that $G \notin \mathcal{O}_{HD}$ by Corollary 3.3.

Having dealt with Case 1, by Observation 3.6 (applied to G^*) it remains to consider the following.

Case 2: there is a finite cut c of G^* such that h_{G^*} witnesses that at least two components of $G^* - c$ are transient.

Consider a minimal subset b of c such that $G^* - b$ still has two transient components D_1 and D_2; this is possible because any supergraph of a transient graph is transient by Theorem 2.3, and so we can let D_1 be one of the transient components of $G^* - c$ and D_2 be the union of the remaining components of $G^* - c$. Then b considered as an edge set of G is the set of edges of a cycle C, such that D_1 and D_2 lie in different sides of C by Lemma 2.1, see Figure 6.

Our plan is to show that the two subgraphs G_1, G_2 of G in either side of C —defined more formally below— are transient, and apply Corollary 3.4 to deduce that $G \notin \mathcal{O}_{HD}$. Since we know that D_1, D_2 are transient subgraphs

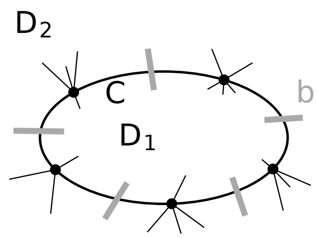

Figure 6: The minimal cut b in G^*, drawn grey, separates the components D_1 and D_2. The corresponding cycle C in G, drawn thick, separates two transient subgraphs associated to the components D_1 and D_2.

of G^*, we would like to pass this information to the dual G to deduce that G_1, G_2 are transient too. The tool we have is Corollary 4.5, but there are two difficulties in applying it: firstly, we need D_1, D_2 to be roundabout-transient rather than just transient to apply this tool. Secondly, D_i is not quite the dual of G_i, as the dual of a subgraph is not quite a subgraph of the dual.

To overcome the first difficulty, recall that every cut of G^* is a cut of $G^{*\circ}$ by the definitions, and so we can think of b as a cut of $G^{*\circ}$. Recall moreover that h_{G^*} was obtained from h by restriction. But since h_{G^*} witnesses that both components D_1, D_2 of $G^* - b$ are transient, it follows from the definitions that h witnesses that both components D'_1, D'_2 of $G^{*\circ} - b$ are transient. In other words, D_1, D_2 are both roundabout transient. Hence their duals are transient by Corollary 4.5.

It remains to overcome the second difficulty, namely to explain the relationship between D_i^* and G_i, where we define G_1 to be the subgraph of G spanned by all vertices lying on the cycle C and its inside, and we define G_2 to be the subgraph of G spanned by all vertices lying on the cycle C and its outside. Let G'_i be the graph obtained from G_i by contracting C into a single vertex (we may create parallel edges by this contraction, but this is ok).

By the definition of the dual of a plane graph, deleting an edge in the primal corresponds to contracting the same edge in the dual, and vice-versa [25]. This is still true when the deleted edges disconnect the graph into two compoenents C_1, C_2; in this case, the corresponding contractions in the dual create a cutvertex v, disconnecting it into two components C'_1, C'_2 and the dual of each C_i coincides with the graph spanned by C'_1 and v. Applying this fact in our situation, we observe that D_i^* coincides with G'_i, because

$D_1 \cup D_2$ is obtained from G^* by deleting the edges in b, and so the dual of $D_1 \cup D_2$ is the graph obtained from G by contracting the edges in C.

To summarise, we have proved that G'_1, G'_2 are transient. Hence so are the subgraphs G''_1, G''_2 of G obtained by deleting the contracted vertex from each of G'_1, G'_2 (in other words, the subgraphs of G lying in either side of C). Applying Corollary 3.4 to these subgraphs, we deduce that $G \notin \mathcal{O}_{HD}$ (to be more precise, we apply Corollary 3.4 to $G''_1, G_2 (= G''_2 \cup C)$ to make sure these subgraphs define a cut of G, i.e. they bipartition $V(G)$, but as transiense is preserved by finite modifications, this is straightforward).

□

7 Proof of the main result

We can now prove Theorem 1.1.

Proof. We have already collected enough tools for the case where G^* is locally finite too: in this case, we can apply Lemma 5.1 to deduce that for some graph H obtained from G by contracting a finite connected subgraph, there are divergence free flows f and h in $H^\circ, H^{*\circ}$ respectively intersecting at a single edge. Plugging this into Theorem 6.3 we deduce that $H \notin \mathcal{O}_{HD}$. Since H differs from G in finitely many vertices and edges, we easily obtain —e.g. using Theorem 3.1— that $G \notin \mathcal{O}_{HD}$ as claimed.

Thus it remains to consider the case where G^* is not locally finite, or in other words, where G has faces bounded by infinitely many edges. We will reduce this case to the above, by constructing a supergraph T of G with locally finite dual T^* such that $G \in \mathcal{O}_{HD}$ if and only if $T \in \mathcal{O}_{HD}$.

For this, let us first construct a supergraph G' of G obtained by triangulating every infinite face of G in such a way that each vertex of G receives at most 2 new edges per incident face (any finite number would do in place of 2); this is easy to do[11]. As $V(G)$ is countable, so is the set of newly added edges. Fix an enumeration $(e_n)_{n\in\mathbb{N}}$ of the set of newly added edges, and subdivide e_n by 2^n new vertices. Let T denote the resulting graph.

Note that T is locally finite, and all its face boundaries are finite, hence T^* is locally finite. Its roundabout graph T° has a subgraph T' which can be obtained from G° by subdividing each edge at most twice: we obtain T' by deleting from T° the roundabouts corresponding to the vertices in $V(T) \setminus V(G)$; the subdivisions are due to the newly added edges e_n. By

[11] The resulting graph is not necessarily a triangulation of $\mathbb{R}^2$, as G may have accumulation points of vertices in its embedding.

Theorem 2.3, T° is transient since G° is. As T^* is locally finite, we can prove that $T \notin \mathcal{O}_{HD}$ by the arguments of the first paragraph of this proof.

We now claim that $T \notin \mathcal{O}_{HD}$ implies the desired $G \notin \mathcal{O}_{HD}$. Indeed, this follows from Corollary 1.2 of [12], which states that if a connected graph G is obtained from a connected graph T by deleting a set of edges of finite total conductance, then $T \in \mathcal{O}_{HD}$ if and only if $G \in \mathcal{O}_{HD}$. In our setup all edges have conductance 1, but we can replace each path of length 2^n that we attached to G to obtain T by a single edge of conductance $1/2^n$; by the classical series law (see e.g. [20, Section 2.3]), this modification results in a network T' that is 'equivalent' to T, in particular, $T' \in \mathcal{O}_{HD}$ if and only if $G \in \mathcal{O}_{HD}$. As the sum of these conductances is finite, the aforementioned result applies, and we deduce that $G \notin \mathcal{O}_{HD}$. □

8 Non-amenable graphs

A vertex is in the *neighbourhood* ∂X of some vertex set X if it is not in X but shares an edge with a vertex in X[12]. An infinite graph G is *non-amenable* if there is a constant $\gamma > 0$ such that for every finite vertex set S of G we have $|\partial S| \geq \gamma \cdot |S|$. The supremum of such values for γ is the *Cheeger-constant* $ch(G)$ of G.

Lemma 8.1. *If a locally finite plane graph G is non-amenable, then so is its roundabout graph G°.*

Proof. Let X be a finite vertex set of G°. Let $\overline{X}$ be the set of those vertices of G whose roundabouts contain vertices of X.

We need to show that $|\partial X| \geq \gamma |X|$ for some $\gamma > 0$. The next claim will imply this under the assumption that X is much larger than $\overline{X}$:

$$\text{Less than } 6 \cdot |\overline{X}| \text{ vertices of } X \text{ have all their neighbours in } X. \tag{6}$$

To prove this, let Y be the set of those vertices of X with all their neighbours in X. If $v \in Y$, then the unique vertex of G° that shares an edge of G with v lies in X. Thus $|Y| \leq 2 \cdot |E(\overline{X})|$, where $E(\overline{X})$ denotes the set of edges of G° with both end-vertices in X. As the subgraph $(\overline{X}, E(\overline{X}))$ of G° spanned by X is planar, it has average degree less than 6, and so $|E(\overline{X})| < 3 \cdot |\overline{X}|$. Thus $|Y| < 6 \cdot |\overline{X}|$ as claimed.

[12] With a slight abuse of notation we use the operator ∂ to denote two unrelated concepts: the difference operator of a potential, as well as the set of neighbours of vertex-sets in the context of non-amenability.

Now if $|X| \geq 12 \cdot |\overline{X}|$, then by (6), at least $|X|/2$ vertices of X have a neighbour outside X. As G° has maximum degree three, ∂X then has size at least $|X|/6$, which fulfils our aim with $\gamma = 1/6$.

Hence it suffices to consider sets X with $|X| < 12 \cdot |\overline{X}|$, and we will assume this is true from now on.

It is reasonable to expect that non-amenability is most difficult to prove when the set X is a union of roundabouts. With this intuition in mind, it is natural to consider the following set. Let $\overline{\overline{X}}$ be the set of those vertices of $\overline{X}$, the whole roundabout of which is in X. Let ϵ be the proportion of the remaining vertices of $\overline{X}$, that is, $\epsilon := (|\overline{X}| - |\overline{\overline{X}}|)/|\overline{X}|$. Our next claim is

$$|\partial X| > \tfrac{\epsilon}{12}|X|. \tag{7}$$

To see this, note that the roundabout x° of each $x \in \overline{X} \setminus \overline{\overline{X}}$ contains a distinct vertex of ∂X, namely, a vertex contained in x° but not in X, hence $|\partial X| \geq |\overline{X} \setminus \overline{\overline{X}}| = \epsilon \cdot |\overline{X}|$. Thus the claim follows from our assumption that $|X| < 12 \cdot |\overline{X}|$.

If ϵ is bounded below, then (7) says that G° is non-amenable. Our next claim will help deal with the case where ϵ is small.

$$|\partial X| \geq K(\epsilon) \cdot |X|, \text{ where } K(\epsilon) = \tfrac{ch(G)\cdot(1-\epsilon)-\epsilon}{12}. \tag{8}$$

Indeed, a lower bound for the neighbourhood ∂X of X is the cardinality of the set $\overline{N}$ of roundabouts containing vertices of ∂X. Clearly, a vertex x of the neighbourhood $\partial\overline{\overline{X}}$ of $\overline{\overline{X}}$ is in $\overline{N}$ unless it is in $\overline{X}$. As x cannot be in $\overline{\overline{X}}$ we can strengthen this statement slightly by replacing $\overline{X}$ by $\overline{X} \setminus \overline{\overline{X}}$. Putting these observations together, we have

$$\begin{aligned} |\partial X| \geq |\overline{N}| &\geq |\partial\overline{\overline{X}}| - |\overline{X} \setminus \overline{\overline{X}}| \\ &\geq ch(G) \cdot |\overline{\overline{X}}| - \epsilon|\overline{X}| \end{aligned}$$

Note that $|\overline{\overline{X}}| = (1-\epsilon) \cdot |\overline{X}|$ by the definition of ϵ. Since we are assuming that $|\overline{X}| > |X|/12$, we obtain the desired $|\partial X| \geq \frac{ch(G)(1-\epsilon)-\epsilon}{12} \cdot |X|$.

Combining (8) with (7) it is straightforward to check that $|\partial X| \geq \gamma|X|$ for some $\gamma > 0$ depending on $ch(G)$. Thus G° is non-amenable. $\square$

We can now prove one of the main results mentioned in the Introduction.

Proof of Theorem 1.4. If G is non-amenable, then so is G° by Lemma 8.1. Every non-amenable locally finite graph is transient as it contains a subtree with positive Cheeger-constant by a result of Benjamini and Schramm [8], and applying this fact to G° proves the statement. □

Combining Theorem 1.4 with Theorem 1.1 immediately yields

Corollary 8.2. *Every locally finite planar non-amenable graph G admits a non-constant Dirichlet harmonic function.*

9 Degree-weighted energy

We define the *degree-weighted energy* $\mathcal{E}_{deg}(f)$ of a flow f in a graph G to be $\sum_{v\in V(G)} deg(v) \left(\sum_{e\ni v} |f(e)|\right)^2$.

Corollary 9.1. *Let G be a locally finite planar graph that admits a flow f from some vertex v such that $\mathcal{E}_{deg}(f)$ is finite. Then G admits a non-constant Dirichlet harmonic function.*

Proof. By Theorem 1.1, it suffices to show that G° is transient. Towards this aim, we extend the flow f on G to a flow g on G° from some vertex v' in the roundabout of v of finite (Dirichlet) energy by assigning values to the edges of the roundabouts.

For a vertex z of G°, we denote by e_z the unique edge of z not in any roundabout. At each roundabout w° of a vertex $w \neq v$ of G, we have to solve a finite Dirichlet-Problem: we want to find a function g_w assigning values to the edges of w° such that at each vertex $z \in w^\circ$, the superimposition of g_w with f satisfies Kirchhoff's node law at all vertices of w°. As f satisfies Kirchhoff's node law at w, it is easy to see that such a g_w always exists, and it is unique up to adding a multiple of the constant flow around w°. Similarly, we can define a function g_v on the edges of v° such that the superimposition of g_v with f satisfies Kirchhoff's node law at all vertices of v° except at a single vertex v' of v°, since f does not satisfy Kirchhoff's node law at v.

We may assume without loss of generality that these g_w satisfy

$$|g_w(k)| \leq \textstyle\sum_{e\ni w} |f(e)| \text{ for every edge } k \text{ of } w^\circ, \tag{9}$$

since otherwise we can add a constant flow of intensity $\sum_{e\ni w} |f(e)|$ around w° to decrease all values of g_w; indeed, this is possible because $|g_w(k) - g_w(k')| \leq \sum_{e\ni w} |f(e)|$ holds for every two edges k, k' of w° by the definition of g_w.

Superimposing f with all the g_x's defines a flow g from v' on G°. By (9), the energy of g is bounded, up to a constant depending on g_v, by $\mathcal{E}(f) + \sum_{v \in V(G)} deg(v) \left(\sum_{e \ni v} |f(e)|\right)^2$, hence it is finite by our assumption (where we also used the fact that $\mathcal{E}_{deg}(f) < \infty$ implies $\mathcal{E}(f) < \infty$ by the definitions). Thus G° is transient by Theorem 2.3. □

Given a locally finite graph G, for an edge $e = vw$ we let $r(e) = deg(v)^2 + deg(w)^2$. We say that G is *super transient* if there is a flow from some vertex with finite r-weighted energy, that is, $\sum_{e \in E(G)} f(e)^2 r(e) < \infty$. Note that super transience implies transience. Moreover, G is super transient if and only if the graph $G[r]$, obtained from G by replacing each edge e with a path of length $r(e)$, is transient.

Corollary 9.2. *Every super transient planar locally finite graph G has a non-constant Dirichlet harmonic function.*

Proof. By the Cauchy-Schwarz inequality, $\left(\sum_{e \ni v} |f(e)|\right)^2 \leq deg(v) \sum_{e \ni v} f(e)^2$. Thus this follows from Corollary 9.1 □

We remark that if we omit the assumption of planarity, then Corollaries 9.1 and Corollary 9.2 become false as the example of the 3-dimensional grid $\mathbb{Z}^3$ shows. The next example shows that Corollary 9.1 is tight in one more sense.

Example 9.3. We construct a locally finite planar graph $G \in \mathcal{O}_{HD}$ admitting a flow f from some vertex such that for every $\epsilon > 0$, we have
$\mathcal{E}_\epsilon(f) = \sum_{v \in V(G)} deg(v)^{(1-\epsilon)} \left(\sum_{e \ni v} |f(e)|\right)^2 < \infty$.

In this construction, we rely on the fact that the 2-dimensional grid $\mathbb{Z}^2$ contains a subdivision T of the infinite binary tree T_2 such that edges at level n are subdivided at most 2^n-times. It is straightforward to construct this subdivision T recursively and we leave the details to the reader. We obtain G from $\mathbb{Z}^2$ by contracting for each edge e of T all but one of its subdivision edges.

By construction, both G and its dual G^* are 1-ended. Moreover, G^* is obtained from $\mathbb{Z}^2$ by deleting edges (again, we are using the fact that deleting an edge in a plane graph corresponds to contracting the same edge in the dual, and vice-versa [25]). Thus by Theorem 6.1, $G \in \mathcal{O}_{HD}$.

Next, we construct f. Let S be the subtree of G consisting of those edges of T that are not contracted. By construction, the tree S is isomorphic to T_2. Let f be the flow on T_2 which assigns edges at level n the value 2^{-n}. Thus f is a flow on G with support S.

Let us estimate $\mathcal{E}_\epsilon(f)$. A vertex v at level n of S has degree at most $8 \cdot 2^n$. Thus

$$\mathcal{E}_\epsilon(f) \leq 1000 \cdot \sum_{n\in\mathbb{N}} 2^n \cdot 2^{n(1-\epsilon)} \cdot 2^{-2n} = 1000 \cdot \sum_{n\in\mathbb{N}} 2^{-\epsilon n}.$$

Hence $\mathcal{E}_\epsilon(f)$ is finite, completing this example.

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